

Log canonical thresholds & Coregularity

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$$(X, B) \text{ lc} \quad I \subseteq \mathbb{R}_{\geq 0} \quad (I \subseteq [0, 1])$$
$$(x, B), \sqcap \geq 0$$

$$lc_t(X, B; \sqcap) = \sup \{ t \in \mathbb{R} \mid (X, B + t \sqcap) \text{ is lc} \}.$$

$$LCT^{\dim=n}(I, J) = \left\{ \underset{\dim=n}{\underset{\downarrow}{lc}} \left(\underset{I}{\frac{x}{n}}, \underset{J}{\frac{B}{n}}; \underset{\sqcap}{\frac{\sqcap}{n}} \right) \right\}.$$

Theorem (HMX'12): Fixing n, I, J satisfying DCC

$LCT^{\dim=n}(I, J)$ satisfies the ACC.

Coregularity (X, B) $\dim X = n$.

$\text{reg}(X, B) = \dim \underline{\mathcal{D}}(Y, B_Y)$ (Y, B_Y) dlt mod values $\{-1, 0, \dots, n-1\}$.

$\lfloor B_Y \rfloor$

flat case. $\text{reg} = -1$.

$\text{coreg}(X, B) = \dim X - \text{reg}(X, B) - \frac{1}{n}$.
values $\{0, \dots, n\}$.

klt case $\text{coreg} = n$.

toric has $\text{coreg} = 0$.

$\text{coreg}(X, B)$ is c : F & minimal log canonical center
 \exists minimal dlt center of $\dim c$ mapping onto it.

For $Z \subset X$ irred subvar.

$\text{reg}(X, B; n_Z) = \text{reg}(U, B|_U)$

U small open neighborhood.

Absolute regularity

$\hat{\text{reg}}(X, B) = \max \left\{ \dim \underline{\mathcal{D}}(X, B + \Gamma) \mid (X, B + \Gamma) \right\}$
log CY.
i.e. $K_X + B + \Gamma = 0$

in the local case we consider l.c.

Coregularity in birational geometry

Lemma A] (X, B) a log C.Y. pair.

S the normalization of a comp. of $\lceil B \rceil$.

$$\text{coreg}(S, B_S) = \text{coreg}(X, B)$$

obtained by adjunction.

Lemma B] Let (X, B) and $X \rightarrow Z$ a fibration,

if every lcc (X, B) dominates Z .

F is a general fiber.

$$\text{then } \text{coreg}(F, B_F) \leq \text{coreg}(X, B) - \dim Z.$$

Lemma C] (X, B) C.Y.

$X \dashrightarrow X'$, then

$$\text{coreg}(X, B) = \text{coreg}(X', B')$$

Coefficients

$$I^+, I^{\pm} \subseteq \log V \left\{ i \in \{0,1\} \mid \sum_{\substack{j \in \\ \text{finite}}} j_k \quad j_k \in I \right\}$$

$$D(I) = \left\{ \alpha \leq 1 \mid \alpha = \frac{m-1+f}{m}, \quad m \in \mathbb{Z}_{\geq 0}, \quad f \in I^+ \right\}$$

$$D_d(I) = \left\{ \alpha \leq 1 \mid \alpha = \frac{m-1+f+kd}{m}, \quad m \in \mathbb{Z}_{\geq 0}, \quad f \in I^+, \quad k \in \mathbb{Z}_{\geq 0} \right\}$$

[Lemma] I satisfying DCC. then.

$I^+, D(I)$ satisfying DCC.

[Lemma] $D(D(I)) = D(I) \cup \{1\}$.

For $d_1 \in D_d(I)$

$$D_{d_1}(I) \subseteq D_d(I).$$

Lemma D \exists I satisfying DCC.

- (X, B) dlt pair. $\text{coeff}(B) \subseteq I$.
- d is a coeff of B . $\underline{\text{coeff}_D B = d}$

Let $S \subseteq LB_I$, with $S \cap D \neq \emptyset$.

(S, B_S) satisfies:

- $\text{coeff}(B_S) \subseteq D(I)$
- For Q of $D \cap S$ $\underline{\text{coeff}_D(B_S)} \in \underline{D_d(I)}$

Lemma E $\exists (X, B)$. D divisor s.t.

$\text{Supp}(D) \cap \text{lcc}(X, B) \neq \emptyset$.

Let (Y, B_Y) be a dlt mod. of (X, B)

then $D_Y \cap LB_{Y'} \neq \emptyset$

Global ACC

Theorem 1] Fix c, I, DCC .

Then there exists $I_0 \subseteq I$ finite s.t.

for (X, B) :

- log canonical
- $\text{coeff}(B) \in I$
- $K_X + B \equiv 0$
- proj. with $\text{wreg}(X, B) \leq c$

Then $\text{coeff}(B) \subseteq I_0$.



For \dim it
is $H^m X$.

Lemma 1] (X, B) log C.Y. with I DCC

- $\text{coeff}(B) \subseteq I$.
- $\text{wreg}(X, B) \leq c$, X proj.

If $d \notin \text{coeff}(B)$. We can construct.

(Z, B_Z) log C.Y. with dimension $\leq c+1$
($\dim = c$ or $Z = \mathbb{P}^1$).

$\text{coeff}(B_Z) \subset D(I)$

there is a coefficient in $D_d(I)$ in B_Z .

Proof of Lemma 1.

If (X, B) is klt, we are done.

We can go inductively.

dlt mod. $(Y, B_Y) \rightarrow (X, B)$.

$S_Y := \text{comp. } LB_Y \lceil$

$D_Y := \text{divisor of } B_Y \text{ with coeff d.}$

For $\varepsilon > 0$ small. $(\underbrace{K_Y + B_Y - \varepsilon S_Y}_{=0}) - \text{MMP.}$

Claim, S_Y is not contracted.

Proof $\phi: Y \rightarrow Y'$. $K_Y + B_Y \equiv 0$.

$$K_Y + B_Y = \phi^* \phi_*(K_Y + B_Y) = \phi^*(K_{Y'} + B_{Y'})$$

$$\alpha_{S_Y}(Y'; B_{Y'}) = 0.$$

$$\varepsilon = \alpha_{S_Y}(Y, \underbrace{B_Y}_{\text{coeff } S_Y B_Y = 1} - \varepsilon S_Y) \leq \alpha_{S_Y}(Y', B_{Y'}) = 0$$

$$\underbrace{K_{Y'} + B_Y - \varepsilon S_Y}_{= 0} \quad \text{not } \underline{\text{ps-eff}}. \quad \text{MMP will term with}$$

MFS.

Case 1 D_Y gets contracted.

$$\pi_1: Y_1 \rightarrow Y_2 \quad \pi_1 \text{ contracts}$$

$(\underbrace{K_{Y_1} + B_1 - \varepsilon S_1}_{\text{-negative curves}})$

i.e., S_1 - positive curves.

such a curve $C \subseteq D_1$, by D_1 being contr.

$$\emptyset \neq C \cap S_1 \subseteq \boxed{D \cap S_1}$$

By our previous Lemmata A, D, E,

we can get (S, B_S) keeping coreg.
 $\dim = 1$.
 $\text{coeff} \subset D(I), D_d(I)$

Case 2 IF D_Y is never contracted.

Y_1 the MFS. has general fiber of $\dim \geq 2$.



• $P(Y_1/Z) = 1 \Rightarrow S_1$ is ample over Z .

as D is effective.

Case 2a) D_1 num. tr.v. over the base. (also $\dim = 1$)

Claim: $D_1 \supseteq C$ curve that contracts to a point.

Proof: $P \in \phi(D_1)$, $C \subseteq \phi^{-1}(P)$.

$$\Rightarrow C \cap D_1 \neq \emptyset, D_1 \cdot C = 0$$

$$\Rightarrow C \subseteq D_1, C \cdot S_1 > 0$$

$$\Rightarrow \underbrace{\phi(C \cap S_1)}_{\phi \neq C \cap S_1} \subseteq S_1 \cap D_1$$

Case 2b) D_1 ample over Z .

F . $D_1|_F, S_1|_F$.

$\dim(D|_F) \geq 1$. we can find $C \subset D_1|_F$.

$S_1|_F$: ample $\Rightarrow C \cdot S_1|_F > 0$

$$\Rightarrow \phi \neq C \cap S_1 \subseteq \boxed{D_1 \cap S_1}$$

Case 3) D is not contracted, and q. fiber of $\dim = 1$.

If D_1 is ample over Z .

We can restrict to Fiber.

we get $(\mathbb{P}^1, B_{\mathbb{P}^1})$ with the conditions,

Proof of Theorem 1] I_0 does not exist

Sketch use Lemma 1. $\Rightarrow d_i \nearrow$ in some family

$$\text{wrg } (x_i, B_i) \leq c.$$

$$\hookrightarrow \dim (Z_i, B'_i) \leq c. \text{ coeff } d_i$$

still appear as $D_d(I)$.

: $f d_i \nearrow$ then these

$d'_i \in D_d(I)$ will also be inf.
many ($\Rightarrow \dim = d$) case.

Global to local.

Theorem? \exists $I \in DCC$.

\exists finite $I_0 \subseteq I$ s.t.

- X normal q.p.
- (X, B) l.c. of $\text{wreg}(X, B) \leq c$
- $\text{coeff}(B) \subseteq I$
- \exists min. lcc $Z \subset X$ of (X, B) contained in every comp. of B .

$\Rightarrow \text{coeff}(B) \subseteq I_0$.

Proof Call $J \subseteq I$ the fin. set of Theorem 1.

$$I_1 := \left\{ i \in I \mid \frac{m^{-1+k_i+f}}{m} \in J \cap [0, 1] \right\}$$

$m, k \in \mathbb{Z}_{\geq 0}$
 $f \in I^+$

set of $i \in I$. s.t. $D_i(I) \cap J \cap [0, 1] \neq \emptyset$

• I_1 is finite.

Claim. is that

$$I_0 := I_1 \cup \{1\}$$

we want

$$D_i(I) \cap J \cap [0, 1] \neq \emptyset$$

i.e. After some adj. we get pairs as in Theorem 1.

$\bullet \overline{(X, B)} \in \mathcal{Z}$, take D any divisor with coeff i .

: If Z is a divisor.

$$\Rightarrow B = Z, \text{ coeff} = 1.$$

Take $\pi: (Y, B_Y) \rightarrow (X, B)$ dlt mod.

Case 1 D_Y intersects S over n_Z .

By adj. (S, B_S) dlt. $\text{coeff} \subseteq B(\mathcal{I})$
 $\text{coeff } D_i(\mathcal{I}).$

Z is minimal lc center.

every lcc of (Y, B_Y) dominates Z .

\Rightarrow For a general fiber (F, B_F) . of $\begin{matrix} S \\ \downarrow \\ Z \end{matrix}$

$\underline{(F, B_F)}$ will be dlt with $K_F + B_F \leq 0$.
and $\text{coeff} \in D_i(\mathcal{I})$.

$\text{coreg}(F, B_F) \leq \text{coreg}(S, B_S) \cdot \underbrace{\dim Z}_{\text{dim } Z}$

\Rightarrow Use Theorem 1

$\Rightarrow \underbrace{D_i(\mathcal{I})}_{\text{by adj.}} \cap \boxed{J} \text{ by Theorem 1}$

Case 2 } $D_Y \wedge S$ does not dominate Z .

We can reduce it to case 1.

\Rightarrow Coeff_{st} still works.

Theorem 3 } ACC for LCT.

c fixed, I, J DCC

$$LCT_c(I, J) := \left\{ \begin{array}{l} lct(X, B; \Gamma) \\ \text{where } \operatorname{wreg}(X, B + t\Gamma) \subseteq \\ \operatorname{coeff}(B) \subseteq I \\ \operatorname{coeff}(\Gamma) \subseteq J \end{array} \right\}$$

satisfies the ACC,

Proof } $t_k = lct(X_k, B_k; \Gamma_k) \nearrow$
 $\forall k$ a min. $lcc(X_k, B_k + t_k \Gamma_k)$
 $V_k \subseteq \operatorname{Supp}(\Gamma_k)$.

$$L := \left\{ i + \frac{t_k}{c} j \right\}$$

$\text{coeff}(B_n + t_n \Gamma) \subseteq L$. also satisfies DCC.

by local ACC. $\exists L_0 \subseteq L$ finite.

$\text{coeff}(B_n + t_n \Gamma) \subseteq L_0$.

$\Rightarrow i_n + \cancel{t_n} \in L_0$ fixed

As $t_n \uparrow$ And $i_n \rightarrow j_n \leftarrow \infty$.

Theorem 1

$$LC(T_0(I, J)) := \left\{ \frac{1-i}{j} \mid \begin{array}{l} i \in I^+ \cap [0, 1] \\ j \in J^+ \setminus 0 \end{array} \right\}$$
